Chapter 22

Categorical Modeling of a State Oriented Version of Linear Logic

Alex Tucker

Abstract. Several systems that promise a logical view of state in programming have been proposed in the literature. We investigate the syntax and semantics of one of these systems called LLMS, developed by Uday Reddy.

We show that the existent categorical definition of a model of LLMS is too conservative and introduce a neat way of revising the definition. We give a concrete construction in a Dialectica category which forms a sound categorical model for the system.

1 Introduction

Since its conception, it was generally assumed that Linear Logic would be able to model state in a straightforward and useful way. While a deal of progress has been made in this direction, see for instance Philip Wadler's emphatically titled paper [Wadler, 1990], the initial hopes for Linear Logic have been tempered somewhat.

In [Reddy, 1993a], Uday Reddy presents a logical system coined LLMS for 'Linear Logic Model of State' which extends Intuitionistic Linear Logic through the introduction of a new, non-commutative tensor product, '$\rhd$', along with an associated modality, '†'. These new constructors, he suggests, have the same relation to each other as the '$\otimes$' and '!' constructors have in Linear Logic. The system was inspired by Christian Retoré's work [Retoré, 1997], which was in turn inspired by Girard's observation that such a non-commutative operator in Linear Logic could be seen as allowing possible one-way communication of resources.

Reddy describes the new tensor as allowing sequential composition of components, while the new modality distinguishes storage objects as being sequentially reusable. He goes on to make these ideas more concrete by showing how these extra structures can be used to model a non-trivial subset of Algol.

Proceedings of the Third ESSLLI Student Session
Ivana Kruijff-Korbayová (editor).

The logical system as presented (see appendix) is syntactically rather messy however — some of the side conditions are quite complex — and we are interested in reformulating it using sound categorical techniques. Retoré already provides constructions in Girard's coherence spaces to model the extra constructions needed for LLMS and Reddy extends this with a description of the necessary categorical constructions.

In this paper we will apply the standard techniques of category theory as a step towards deriving a sound syntactic characterization of the system. To this end, we have found that Valeria De Paiva's Dialectica Categories [de Paiva, 1989], which already provide a class of models for Linear Logic, also allow for a non-commutative tensor and as such are directly applicable as a semantic model for LLMS. This connection has already been partially explored in [da S. Corrêa et al., 1996] for a variant of Reddy's system called LLMS_c, with the new tensor relaxed to be commutative, motivated by considering it as an interleaving operator, "representing concurrent execution without interaction or synchronization."

We will show that initial thoughts on how to model 'serialization' in the system are too conservative and introduce a revised model that completely captures this notion in a neat way.

2 LLMS Categories

Reddy uses coherence spaces as a concrete model for LLMS, but he also sets out the criteria he believes are needed for a categorical model, which we will use as a guide. He starts with a symmetric monoidal closed category $(\mathcal{C}, \otimes, \mathbf{I}, \multimap)$ together with an additional monoidal structure $(\rhd, \mathbf{I})$ such that $(\otimes, \mathbf{I})$ is a sub-monoidal structure of $(\rhd, \mathbf{I})$. The last condition means that there is a natural monic, $\mathbf{ser}\colon A \otimes B \to A \rhd B$, which preserves the associated monoidal structure.

Using this, he goes on to require two monoidal comonads, $!$ and $\dagger$, with the former being a sub-comonad of the latter via a comonad monomorphism, $\mathbf{Ser}\colon ! \to \dagger$. Also, $!A$ must be a comonoid with respect to $(\otimes, \mathbf{I})$ and $\dagger A$ a comonoid with respect to $(\rhd, \mathbf{I})$. Then, since $(\otimes, \mathbf{I})$ is a sub-monoidal structure of $(\rhd, \mathbf{I})$, $!A$ becomes a comonoid with respect to $(\rhd, \mathbf{I})$ via the monic $\mathbf{ser}$. This comonoid must be a 'sub-comonoid' of $\dagger A$ via the natural monic $\mathbf{Ser}$ which must now be a morphism of comonoids as well.

Reddy then adds the usual requirements, due to Seely, that the category have finite products and coproducts, and that $!$ should map the product monoidal structure $(\&, \top)$ to $(\otimes, \mathbf{I})$. He calls the resulting structure an *LLMS-category.*

2.1 Complexities of the Ser Rule

We have found that this definition of an LLMS-category is not quite enough for a model of the LLMS calculus however. More specifically, the following logical rule, called **Ser**[1], leads to some problems.

Recall that contexts in LLMS are built using two operators: ‘,’ and ‘;’. In the rule above, |, | is simply the (multi)set of formula occurrences in , and $\leq_,$ is a partial order on , defined as follows:

$$\begin{array}{rcl} \leq_A &=& \{(A,A)\} \\ \leq_{,\;,\Delta} &=& \leq_, \cup \leq_\Delta \\ \leq_{,\;;\Delta} &=& \leq_{,\;,\Delta} \cup\ |,\ | \times |\Delta| \end{array} \qquad (22.1)$$

The following transformations are direct consequences of rule (**??**):

$$\begin{array}{rl} \mathbf{ser}: & A\otimes B \to A \rhd B \\ \mathbf{wd1}: & A\otimes (B\rhd C) \to (A\otimes B)\rhd C \\ \mathbf{wd2}: & (A\rhd B)\otimes C \to A\rhd (B\otimes C) \\ \mathbf{int}: & (A\rhd B)\otimes (C\rhd D) \to (A\otimes C)\rhd (B\otimes D) \end{array}$$

They all draw on the fact that we can impose the same order on formulæ as we can on contexts, by turning ⊗’s into commas, ▷’s into semicolons and applying ≤ to the resulting context. A formula on the left can then be transformed into the one on the right if the order on the left formula is included in the order on the right formula. The following derivation of **wd1** will make this clearer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A\,,B \vdash A\otimes B}\otimes\mathcal{R} \quad C\vdash C}{(A\,,B)\,;C \vdash (A\otimes B)\rhd C}\rhd\mathcal{R}}{A\,,(B\,;C)\vdash (A\otimes B)\rhd C}\mathbf{Ser}}{A\,,(B\rhd C)\vdash (A\otimes B)\rhd C}\rhd\mathcal{L}}{A\otimes(B\rhd C)\vdash(A\otimes B)\rhd C}\otimes\mathcal{L}$$

Using the same argument, rule (**??**) also induces commutativity and associativity of ‘⊗’ and associativity of ‘▷’.

---

[1] The unfortunate profusion of things called ‘Ser’ becomes confusing in the following, so we will be more explicit when possible.

The first three of these transformations, corresponding to the existent natural transformation, **ser**, and two forms of weak distributivity, were pointed out by Reddy in [Reddy, 1993b]. The fourth — one half of the middle interchange law — was found to be necessary for a model of $\mathrm{LLMS_c}$ by Corrêa.

Contrary to what we originally expected however, the transformation **ser** is not strictly strong enough to derive the other three transformations in 23.1, even with the various associativity and commutativity conditions on '$\otimes$' and '$\rhd$'.

2.2 Series Parallel Orders

The partial order over the contexts of the LLMS calculus are in fact a class of order called 'series parallel orders' (SP-orders). These are defined as the least class of partial orders containing the single element order, and closed under disjoint union and ordinal sum.

In [Bechet et al., 1997], a set of rewrite rules is given which is a *complete* characterization of the inclusion of one SP-order in another and so by extension corresponds exactly to Reddy's 'Ser' rule.

In fact, the non-trivial rewrite rules correspond precisely to the four rules given in 23.1 above. Moreover, if we insist that both '$\otimes$' and '$\rhd$' have identities which are equal, that '$\otimes$' is associative and commutative and that '$\rhd$' is associative and *not* commutative, then the set of rewrite rules are all derivable from just the middle interchange rule.

The transformation **ser** can be obtained from **int** by instantiating B and C as $\mathbf{I}$; **wd1** can be obtained by instantiating B as $\mathbf{I}$ and **wd2** by instantiating C as $\mathbf{I}$. We therefore need to exchange **ser** for **int** in the definition of an LLMS-category.

Categorically speaking, requiring a natural transformation **int** is quite reasonable. Considering $\rhd$ as monoidal over $\otimes$ we get $\rhd : \mathcal{C}^2(\otimes^2) \to \mathcal{C}(\otimes)$ where $\otimes^2$ is just the pairwise construction of $\otimes$. For this to be monoidal means, for one, that:

$$\rhd(A,B) \otimes \rhd(C,D) \to \rhd((A,B) \otimes^2 (C,D))$$

which written more usually with $\rhd$ as an infix operator and $\otimes^2$ expanded, gives:

$$(A \rhd B) \otimes (C \rhd D) \to (A \otimes C) \rhd (B \otimes D)$$

2.3 A Revised Definition

We are now in a position to state a more precise definition of an LLMS-category, which we will build in two stages over a modality free variant. We will relax Reddy's requirements somewhat by neither insisting on monicity of the natural transformations, nor dealing with the additive constructs. We also exchange the natural transformation **ser** with a natural transformation **int** corresponding to the middle interchange rule above, hence:

Definition 1 *A* modality-free LLMS-category *is an SMCC,* $(\mathcal{C}, \otimes, \mathbf{I}, \multimap)$ *together with an additional monoidal structure* $(\rhd, \mathbf{I})$*. The two structures are related via a monoidal natural transformation,* ***int***: $(A \rhd B) \otimes (C \rhd D) \to (A \otimes C) \rhd (B \otimes D)$.

An LLMS-category *is then a modality-free LLMS-category along with two monoidal comonads* $!$ *and* $\dagger$ *related by a comonad morphism,* ***Ser***: $! \to \dagger$. *Also,* $!A$ *must be a comonoid with respect to* $(\otimes, \mathbf{I})$*,* $\dagger A$ *a comonoid with respect to* $(\rhd, \mathbf{I})$ *and* ***Ser*** *a morphism of comonoids.*

3 A Dialectica Model for LLMS

The Dialectica Categories [de Paiva, 1989] spring from a categorical formulation of Gödel's Dialectica Interpretation of higher order arithmetic. Amongst other more general models, they also give rise to a model of Intuitionistic Linear Logic. The categories allow for different types of tensor, including non-commutative tensors, so we will take advantage of their versatility.

Our categorical model of LLMS is based on a simple Dialectica Category over the category of sets, where the objects are partial relations over a three valued poset, or maps into $\mathbf{3} = \{\mathsf{false}, \mathsf{undefined}, \mathsf{true}\}$.

Definition 2 *Let* $\mathcal{L}$ *be a category whose:*

- *Objects are partial relations on* ***Set****. An object* A *given by the partial relation* α *from the sets* U *and* X*,* $\alpha \colon U \times X \to \mathbf{3}$*, will be depicted as* $A = (U \overset{\alpha}{\leftarrow\!\!\!\!+} X)$*. We write relations using the infix notation, for example, '*$u\,\alpha\,x$*', '*$u\,\neg\alpha\,x$*' or '*$u\,\alpha\,x$ *is undefined'.*

- *Morphisms are pairs of functions* $(f, F) \colon (U \overset{\alpha}{\leftarrow\!\!\!\!+} X) \to (V \overset{\beta}{\leftarrow\!\!\!\!+} Y)$*, where* $f \colon U \to X$ *and* $F \colon Y \to V$*, satisfying the condition that, for all* $u \in U$ *and* $y \in Y$*,* $u \alpha F(y)$ *implies* $f(u) \beta y$*. We will depict morphisms thus:*

$$\begin{tikzcd}
U & X \arrow[l, "\alpha"', "|" marking] \\
V \arrow[from=1-1, "f"'] & Y \arrow[l, "\beta"', "|" marking] \arrow[u, "F"']
\end{tikzcd}$$

- *Composition of morphisms* (f, F) *and* (g, G) *is composition in each coordinate:* $(f, F) ; (g, G) = (f ; g, G ; F)$*. Identity morphisms are pairs of identities,* (Id_U, Id_X)*.*

We can extend the category $\mathcal{L}$ with a tensor bifunctor $-\otimes-\colon \mathcal{L}\times\mathcal{L}\to\mathcal{L}$ and internal-hom bifunctor $[-,-]\colon \mathcal{L}^{\mathrm{op}}\times\mathcal{L}\to\mathcal{L}$, defined as follows:

Definition 3 *For objects,* $A = (U \overset{\alpha}{\leftarrow\!\!\!+} X)$ *and* $B = (V \overset{\beta}{\leftarrow\!\!\!+} Y)$*, there is a tensor bifunctor and an internal-hom bifunctor. The tensor is given by:*

$$A\otimes B = (U\times V \overset{\alpha\otimes\beta}{\leftarrow\!\!\!+} X^V \times Y^U)$$

where $(u,v)\;\alpha\otimes\beta\;(f,g)$ *iff* $u\;\alpha\;f(v)$ *and* $v\;\beta\;g(u)$*. Its unit is the object* $\mathbf{I} = (\mathbf{1} \overset{\iota}{\leftarrow\!\!\!+} \mathbf{1})$*, where* ι *is the identity relation on* $\mathbf{1}$*. The internal-hom is given by:*

$$[A,B]_{\mathcal{L}} = (V^U \times X^Y \overset{\beta^\alpha}{\leftarrow\!\!\!+} U\times Y)$$

where $(f,F)\;\beta^\alpha\;(u,y)$ *iff* $u\;\alpha\;F(y)$ *implies* $f(u)\;\beta\;y$*.*

It can be shown that the above really does define a category, and that the tensor is monoidal, commutative and adjoint to the internal-hom, so that:

Lemma 1 *The structure* $(\mathcal{L},\otimes,\mathbf{I},\multimap)$ *is a symmetric monoidal closed category.*

A non-commutative tensor bifunctor, $-\rhd-\colon \mathcal{L}\times\mathcal{L}\to\mathcal{L}$ can be defined in a similar way, with the same unit as $\otimes$ above.

Definition 4 *There is a non-commutative tensor bifunctor,*

$$A\rhd B = (U\times V \overset{\alpha\rhd\beta}{\leftarrow\!\!\!+} X\times Y)$$

where $(u,v)\;\alpha\rhd\beta\;(x,y)$ *iff* $u\;\alpha\;x$ *land*[2] $v\;\beta\;y$*. Its unit is the same object* $\mathbf{I}$ *as above.*

Some calculation is necessary to show that this tensor bifunctor is monoidal and non-commutative and that we can exhibit a natural transformation $\mathbf{int}\colon (A\rhd B)\otimes(C\rhd D)\to(A\otimes C)\rhd(B\otimes D)$, so that:

Lemma 2 *The category given by the SMCC* $(\mathcal{L},\otimes,\mathbf{I},\multimap)$ *along with the monoidal bifunctor* $(\rhd,\mathbf{I})$ *and monoidal natural transformation* ***int*** *is a modality-free LLMS-category.*

We now define two endofunctors on $\mathcal{L}$ to model the modalities ! and †.

---

[2] Here 'land' is the computer science 'lazy and'. It is non-commutative in the sense that false land undefined is false, but undefined land false is undefined.

Definition 5 *The functor* $\dagger$ *acts on objects* $A = U \overset{\alpha}{\longleftarrow\!\!\!\!\!\!+\;} X$ *of* $\mathcal{L}$ *as follows:*

$$\dagger A = (U \overset{\dagger\alpha}{\longleftarrow\!\!\!\!\!\!+\;} X^{*})$$

where $u\,(\dagger\alpha)\,(x_1, \ldots, x_k)$ *iff* $u\,\alpha\,x_1$ *and* $\ldots$ *and* $u\,\alpha\,x_k$. *The functor acts on morphisms as follows:*

$$\dagger(f, F) = (f, F^{*})$$

The functor ! *acts on objects as follows:*

$$!A = (U \overset{!\alpha}{\longleftarrow\!\!\!\!\!\!+\;} (X^{*})^{U})$$

where $u\,(!\alpha)\,f$ *iff* $u\,\alpha\,x_1$ *and* $\ldots$ *and* $u\,\alpha\,x_k$, *where* $f(u) = (x_1, \ldots, x_k)$. *The functor acts on morphisms as follows:*

$$!(f, F) = (f, F^{*}(-)f)$$

With the endofunctors so defined, we can show that they are indeed monoidal comonads, that ! is a comonoid with respect to $(\otimes, \mathbf{I})$, $\dagger$ a comonoid with respect to $(\rhd, \mathbf{I})$ and that the two are related to each other by a comonad morphism, **Ser** : $! \to \dagger$. This is enough to show the soundness of the categorical model by the usual method of induction over the derivation. We simply substitute **int** to model the vagaries of the logical rule **Ser**.

Theorem 1 *The category* $\mathcal{L}$, *together with bifunctors* $\otimes$, $[-,-]_{\mathcal{L}}$ *and* $\rhd$, *and modalities* ! *and* $\dagger$ *is a an LLMS-category. This means:*

- *For all proofs* , $\vdash A$ *in the type theory LLMS, there is a morphism,* $(f, F)\colon [\![,]\!] \to [\![A]\!]$, *in* $\mathcal{L}$.
- *For all provable equalities* , $\vdash M = N\colon A$, *we have equality of morphisms in the category* $\mathcal{L}$, $[\![M]\!] =_{\mathcal{L}} [\![N]\!]$.

4 Conclusion

We have shown that a system arising from investigations into sequentiality and state can be modeled neatly as another Dialectica Category. This extends the original work on $\mathrm{LLMS_c}$ in a novel way and enables us to approach the problem of unraveling the syntax of LLMS using categorically sound techniques. The fact that we *can* form a model of state in a Dialectica Category is a testament to the versatility of these constructions.

The obvious next steps are to give a term assignment for this system using the transformation **int** as a term constructor to overcome the problems of the logical rule **Ser**. We will then be able to investigate a completeness theorem for the model. The additive rules in the system will also have to be incorporated if we are to fully substantiate the claim that we can offer an abstract model for Idealized Algol.

More recently, Peter O'Hearn describes in [O'Hearn, 1997] a calculus related to Reddy's system LLMS in its use of 'contexts as trees', which he calls Bunches — a term taken from relevance logic. We would like to further investigate the relationship of Reddy's system to that of O'Hearn's and see if our categorical insights can be brought to play.

Other lines of research involve investigating the various transformations induced by the **Ser** rule. The property of weak distributivity between the two types of tensor occurs often, see for instance work by Cockett and Seeley. The monoidal construction leading to the derivation of **int** has also occured in the literature from work on Petri Nets by José Mesenguer.

We could further apply our insights on the rôle of the **Ser** rule to other classes of model including the original Coherence Space model and perhaps some form of Games model. In fact, the literature already offers a natural interpretation of Games in Dialectica Categories.

Acknowledgments

I would like to thank Valeria de Paiva, Eike Ritter and Neil Ghani for their deep insights and great help in this work.

Bibliography

[Bechet et al., 1997]	Denis Bechet, Philippe de Groote, and Christian Retoré. A complete axiomatisation for the inclusion of series-parallel partial orders. In *Proceedings of the Eighth International Conference on Rewriting Techniques and Applications*, Sitges, Spain, 1997.
[da S. Corrêa et al., 1996]	Marcelo da S. Corrêa, Edward H. Haeusler, and Valeria C.V. de Paiva. A dialectica model of state. In *Proceedings of Computing: The Australasian Theory Symposium (CATS'96)*, January 1996.
[de Paiva, 1989]	Valeria C. V. de Paiva. The dialectica categories. In John W. Gray and Andre Scedrov, editors, *Categories in Computer Science and Logic*, volume 92 of *Contemporary Mathematics*, pages 47–62, Providence, Rhode Island, 1989. American Mathematical Society.
[O'Hearn, 1997]	Peter O'Hearn. Bunches, typed lambda-calculus and interference control. Draft paper, October 1997.
[Reddy, 1993a]	Uday S. Reddy. A linear logic model of state. Electronic manuscript, University of Illinois, October 1993.

[Reddy, 1993b] Uday S. Reddy. A linear logic model of state (extended abstract). Technical report, University of Glasgow, January 1993.

[Retoré, 1997] Christian Retoré. Pomset logic: a non-commutative extension of classical linear logic. In *TLCA*, volume 1210, pages pp300–318. LNCS, 1997.

[Wadler, 1990] Philip Wadler. Linear types can change the world. In *TC 2 Working Conference on Programming Concepts and Methods (Preprint)*, pages 546–566, 1990.

Proof Rules for LLMS

We give the rules of LLMS for reference only, see [Reddy, 1993a] for details. The contexts, , , are ‘bunches’ — that is trees with nodes labeled by the comma and semicolon operators and leaves labeled by formulæ, along with a partial order, $\leq_{,}$ (see above). The notation , [Δ] denotes a context , with a sub-context (subtree) Δ. , is defined to be *independent* if it has no semicolon connectives.

Structural Rules

$$\frac{,\ ' \vdash A}{,\ \vdash A}\ \textsf{Ser}\quad \text{if } |,\ | = |,\ '| \text{ and } \leq_{,} \subseteq \leq_{,'}$$

$$\frac{}{A \vdash A}\ \textsf{Id} \qquad \frac{,\ \vdash A}{\Delta[,\ ] \vdash B}\ \textsf{Cut}$$

$$\Delta[A] \vdash B$$

$$\frac{,\ \vdash A}{,\ ,\Delta \vdash A \otimes B}\ \otimes\mathcal{R}$$

$$\Delta \vdash B \qquad \frac{,\ [A\ ,B] \vdash C}{,\ [A \otimes B] \vdash C}\ \otimes\mathcal{L}$$

$$\frac{,\ \vdash A}{,\ ;\Delta \vdash A \rhd B}\ \rhd\mathcal{R}$$

$$\Delta \vdash B \qquad \frac{,\ [A\ ;B] \vdash C}{,\ [A \rhd B] \vdash C}\ \rhd\mathcal{L}$$

$$\frac{}{\vdash \mathbf{I}}\ \mathbf{I}\mathcal{R} \qquad \frac{,\ [\epsilon] \vdash C}{,\ [\mathbf{I}] \vdash C}\ \mathbf{I}\mathcal{L}$$

$$\frac{,\ ,A \vdash B}{,\ \vdash A \multimap B}\ \multimap\mathcal{R} \qquad \frac{,\ \vdash A}{\Delta[,\ ,A \multimap B] \vdash C}\ \multimap\mathcal{L}$$

$$\Delta[B] \vdash C$$

Multiplicative Rules

Modalities

$$\frac{,\ [\epsilon] \vdash C}{,\ [\dagger A] \vdash C}\ \dagger\textsf{Weak} \qquad \frac{,\ [A] \vdash C}{,\ [\dagger A] \vdash C}\ \dagger\textsf{Der}$$

$$\frac{,\ [\dagger A \rhd \dagger A] \vdash C}{,\ [\dagger A] \vdash C}\ \dagger\mathsf{Thread}$$

$$\frac{,\ \vdash A}{,\ \vdash \dagger A}\ \dagger\mathcal{R}$$ if , is an independent context with only ! or † formulæ

$$\frac{,\ [\dagger A] \vdash C}{,\ [!A] \vdash C}\ !\mathsf{Ser} \qquad \frac{,\ [!A \otimes !A] \vdash C}{,\ [!A] \vdash C}\ !\mathsf{Contr}$$

$$\frac{,\ \vdash A}{,\ \vdash !A}\ !\mathcal{R}$$ if , is an independent context with only ! formulæ