Categorical Modelling of a State Oriented Version of Linear Logic

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1 A Linear Logic Model of State

Since its conception, it was generally assumed that Linear Logic would be able to model state in a straightforward and useful way. While a deal of progress has been made in this direction, see for instance Philip Wadler's emphatically titled paper [Wad90], the initial hopes for Linear Logic have been tempered somewhat.

In [Red93], Uday Reddy presents a logical system coined LLMS for 'Linear Logic Model of State' which extends Intuitionistic Linear Logic by introducing a new, non-commutative tensor product, '$\triangleright$', along with an associated modality, '$\dagger$'. These new constructors, he suggests, have the same relation to each other as the '$\otimes$' and '!' constructors have in Linear Logic. The system was inspired by Christian Retoré's work [Ret97], which was in turn inspired by Girard's observation that such a non-commutative operator in Linear Logic could be seen as allowing possible one-way communication of resources.

Reddy describes the new tensor as allowing sequential composition of components, while the new modality distinguishes storage objects as being sequentially reusable. He goes on to make these ideas more concrete by showing how these extra structures can be used to model a non-trivial subset of Algol.

The logical system as presented (see appendix) is syntactically rather messy however, and we are interested in reformulating it using sound categorical techniques. Retoré already provides constructions in Girard's coherence spaces to model the extra constructions needed for LLMS and Reddy extends this with a description of the necessary categorical constructions.

In this paper we will apply the standard techniques of category theory as a step towards deriving a sound syntactic characterisation of the system. To this end, we have found that Valeria De Paiva's Dialectica Categories [dP89], which already provide a class of models for Linear Logic, also allow for a non-commutative tensor and as such are directly applicable as a semantic model for LLMS. This connection has already been partially explored in [CHdP96] for a variant of Reddy's system called $\mathrm{LLMS_C}$, with the new tensor relaxed to be commutative, motivated by considering it as an interleaving operator, "representing concurrent execution without interaction or synchronisation."

More recently, Peter O'Hearn describes in [O'H97] a calculus related to Reddy's system LLMS in its use of 'contexts as trees', which he calls Bunches. We aim to relate our work to his at a later stage.

2 LLMS Categories

Reddy uses coherence spaces as a concrete model for LLMS, but he also sets out the criteria he believes are needed for a categorical model, which we will use as a guide. He starts with a symmetric monoidal closed category $(\mathbf{C}, \otimes, \mathbf{I}, \multimap)$ together with an additional monoidal structure $(\rhd, \mathbf{I})$ such that $(\otimes, \mathbf{I})$ is a submonoidal structure of $(\rhd, \mathbf{I})$. The last condition means that there is a natural monic, $\mathbf{ser}\colon A \otimes B \to A \rhd B$, which preserves the associated monoidal structure.

Using this, he goes on to require two monoidal comonads, ! and †, with the former being a sub-comonad of the latter via a comonad monomorphism, $\mathbf{Ser}\colon ! \to \dagger$. Also, $!A$ must be a comonoid with respect to $(\otimes, \mathbf{I})$ and $\dagger A$ a comonoid with respect to $(\rhd, \mathbf{I})$. Then, since $(\otimes, \mathbf{I})$ is a submonoidal structure of $(\rhd, \mathbf{I})$, $!A$ becomes a comonoid with respect to $(\rhd, \mathbf{I})$ via the monic **ser**. This comonoid must be a 'subcomonoid' of $\dagger A$ via the natural monic **Ser** which must now be a morphism of comonoids as well.

Reddy then adds the usual requirements, due to Seely, that the category have finite products and coproducts, and that ! should map the product monoidal structure $(\&, \top)$ to $(\otimes, \mathbf{I})$. He calls the resulting structure an *LLMS-category.*

We will relax Reddy's requirements somewhat by neither insisting on monicity of the natural transformations above, nor dealing with the additive constructs, hence:

Definition 1. *An* LLMS-category *consists of an SMCC,* $(\mathbf{C}, \otimes, \boldsymbol{I}, \multimap)$ *together with an additional monoidal structure* $(\rhd, \boldsymbol{I})$*. The two structures are related via a monoidal natural transormation,* $\boldsymbol{ser}\colon A \otimes B \to A \rhd B$*.*

Additionally, $\mathbf{C}$ *has two monoidal comonads* ! *and* † *related by a comonad morphism,* $\boldsymbol{Ser}\colon ! \to \dagger$*. Also,* $!A$ *must be a comonoid with respect to* $(\otimes, \boldsymbol{I})$*,* $\dagger A$ *a comonoid with respect to* $(\rhd, \boldsymbol{I})$ *and* $\boldsymbol{Ser}$ *a morphism of comonoids.*

3 A Dialectica Model for LLMS

The Dialectica Categories [dP89] spring from a categorical formulation of Gödel's Dialectica Interpretation of higher order arithmetic. Our categorical model of LLMS is based on a simple Dialectica Category over the category of sets, where the objects are partial relations over a three valued logic, or maps into $\mathbf{3} = \{\mathsf{false}, \mathsf{undefined}, \mathsf{true}\}$.

Definition 2. *Let $\mathcal{L}$ be a category whose:*

- *Objects are partial relations on* ***Set****. An object A given by the partial relation α from the sets U and X, $\alpha{:}\, U \times X \to \mathbf{3}$, will be depicted as $A = (U \overset{\alpha}{\leftarrow\!\!\!+} X)$. We write relations using the infix notation, for example, $u \, \alpha \, x$, $u \, \neg\alpha \, x$ or $u \, \alpha \, x$ is undefined.*

- *Morphisms are pairs of functions $(f, F){:}\, (U \overset{\alpha}{\leftarrow\!\!\!+} X) \to (V \overset{\beta}{\leftarrow\!\!\!+} Y)$, where $f{:}\, U \to X$ and $F{:}\, Y \to V$, satisfying the condition that, for all $u \in U$ and $y \in Y$, $u \alpha F(y)$ implies $f(u) \beta y$. We will depict morphisms thus:*

$$\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l, "\alpha"', "\shortmid" marking] \\
V & Y \arrow[l, "\beta"', "\shortmid" marking] \arrow[u, "F"']
\end{tikzcd}$$

- *Composition of morphisms (f, F) and (g, G) is composition in each coordinate: $(f, F) \,;(g, G) = (f \,; g, G \,; F)$. Identity morphisms are pairs of identities, (Id_U, Id_X).*

We can extend the category $\mathcal{L}$ with a tensor bifunctor $- \otimes -{:}\, \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and internal-hom bifunctor $[-,-]{:}\, \mathcal{L}^{\mathrm{op}} \times \mathcal{L} \to \mathcal{L}$, defined as follows, which give rise to an SMCC.

Definition 3. *For objects, $A = (U \overset{\alpha}{\leftarrow\!\!\!+} X)$ and $B = (V \overset{\beta}{\leftarrow\!\!\!+} Y)$, there is a tensor bifunctor and an internal-hom bifunctor. The tensor is given by:*

$$A \otimes B = (U \times V \overset{\alpha\otimes\beta}{\leftarrow\!\!\!+} X^V \times Y^U)$$

where $(u, v) \, \alpha \otimes \beta \, (f, g)$ iff $u \, \alpha \, f(v)$ and $v \, \beta \, g(u)$. Its unit is the object $\boldsymbol{I} = (\mathbf{1} \overset{\iota}{\leftarrow\!\!\!+} \mathbf{1})$, where ι is the identity relation on $\mathbf{1}$. The internal-hom is given by:

$$[A, B]_{\mathcal{L}} = (V^U \times X^Y \overset{\beta^\alpha}{\leftarrow\!\!\!+} U \times Y)$$

where $(f, F) \, \beta^\alpha \, (u, y)$ iff $u \, \alpha \, x$ implies $v \, \beta \, y$.

A new tensor, $- \rhd -{:}\, \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ can be defined in a similar way, and we must show that it is indeed non-commutative with the same unit as $\otimes$.

Definition 4. *There is a non-commutative tensor bifunctor,*

$$A \rhd B = (U \times V \overset{\alpha\rhd\beta}{\leftarrow\!\!\!+} X \times Y)$$

where $(u, v)\ \alpha \rhd \beta\ (x, y)$ *iff* $u\,\alpha\,x$ *land*[1] $v\,\beta\,y$. *Its unit is the same object* $\boldsymbol{I}$ *as above.*

We now define two endofunctors on $\mathcal{L}$ to model the modalities ! and †.

Definition 5. *The functor* † *acts on objects* $A = U \overset{\alpha}{\longleftarrow\!\!\!\!+} X$ *of* $\mathcal{L}$ *as follows:*

$$\dagger A = (U \overset{\dagger\alpha}{\longleftarrow\!\!\!\!+} X^*)$$

where $u\ (\dagger\alpha)\ (x_1, \ldots, x_k)$ *iff* $u\,\alpha\,x_1$ *and* … *and* $u\,\alpha\,x_k$. *The functor acts on morphisms as follows:*

$$\dagger(f, F) = (f, F^*)$$

The functor ! *acts on objects as follows:*

$$!A = (U \overset{!\alpha}{\longleftarrow\!\!\!\!+} (X^*)^U)$$

where $u\ (!\alpha)\ f$ *iff* $u\,\alpha\,x_1$ *and* … *and* $u\,\alpha\,x_k$, *where* $f(u) = (x_1, \ldots, x_k)$. *The functor acts on morphisms as follows:*

$$!(f, F) = (f, F^*(-)f)$$

Theorem 1. *The category* $\mathcal{L}$, *together with bifunctors* $\otimes$, $[-,-]_{\mathcal{L}}$ *and* $\rhd$, *and modalities* ! *and* † *is a an LLMS-category. This means:*

- *For all proofs* $, \vdash A$ *in the type theory LLMS, there is a morphism,* $(f, F)\colon [\![,]\!] \to [\![A]\!]$, *in* $\mathcal{L}$.
- *For all provable equalities* $, \vdash M = N\colon A$, *we have equality of morphisms in the category* $\mathcal{L}$, $[\![M]\!] =_{\mathcal{L}} [\![N]\!]$.

4 Conclusion

We have shown that a system arising from investigations into sequentiality and state can be modelled neatly as another Dialectica Category. This extends the original work on $\mathrm{LLMS_C}$, and enables us to approach the problem of unravelling the syntax of LLMS in a more general, yet theoretically sound way. The fact that we *can* form a model of state in a Dialectica Category is a testament to the versatility of these constructions.

We would like to further investigate the relationship of Reddy's system to that of O'Hearn's and see if our categorical insights can be brought to play. Another line of investigation is due to the property of weak distributivity between the two types of tensor, in that this property also arises quite naturally in, for instance, work by Cockett and Seeley.

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[1] Here 'land' is the computer science 'lazy and'. It is non-commutative in the sense that false land undefined is false, but undefined land false is undefined.

References

[CHdP96] Marcelo da S. Corrêa, Edward H. Haeusler, and Valeria C.V. de Paiva. A dialectica model of state. In *Proceedings of Computing: The Australasian Theory Symposium (CATS'96)*, January 1996.

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[O'H97] Peter O'Hearn. Bunches, typed lambda-calculus and interference control. Draft paper, October 1997.

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[Wad90] Philip Wadler. Linear types can change the world. In *TC 2 Working Conference on Programming Concepts and Methods (Preprint)*, pages 546–566, 1990.

Proof Rules for LLMS

We give the rules of LLMS for reference only, see [Red93] for details. The contexts, , , are 'bunches' — that is trees with branches formed by the comma and semicolon operators along with a partial order, $\leq_\Gamma$ forcing certain associativity and commutativity equivalences. , is *independent* if it has no semicolon connectives.

Structural Rules

$$\frac{,' \vdash A}{, \vdash A}\ \mathsf{Ser}\quad \text{if } |,| = |,'| \text{ and } \leq_\Gamma \subseteq \leq_{\Gamma'}$$

$$\frac{}{A \vdash A}\ \mathsf{Id} \qquad \frac{, \vdash A \quad \Delta[A] \vdash B}{\Delta[,] \vdash B}\ \mathsf{Cut}$$

Multiplicative Rules

$$\frac{, \vdash A \quad \Delta \vdash B}{, , \Delta \vdash A \otimes B}\ \otimes\mathcal{R} \qquad \frac{, [A , B] \vdash C}{, [A \otimes B] \vdash C}\ \otimes\mathcal{L}$$

$$\frac{, \vdash A \quad \Delta \vdash B}{, ; \Delta \vdash A \rhd B}\ \rhd\mathcal{R} \qquad \frac{, [A ; B] \vdash C}{, [A \rhd B] \vdash C}\ \rhd\mathcal{L}$$

$$\frac{}{\vdash \mathbf{I}}\ \mathbf{I}\mathcal{R} \qquad \frac{, [\epsilon] \vdash C}{, [\mathbf{I}] \vdash C}\ \mathbf{I}\mathcal{L}$$

$$\frac{, , A \vdash B}{, \vdash A \multimap B}\ \multimap\mathcal{R} \qquad \frac{, \vdash A \quad \Delta[B] \vdash C}{\Delta[, , A \multimap B] \vdash C}\ \multimap\mathcal{L}$$

Modalities

$$\frac{, [\epsilon] \vdash C}{, [\dagger A] \vdash C}\ \dagger\mathsf{Weak} \qquad \frac{, [A] \vdash C}{, [\dagger A] \vdash C}\ \dagger\mathsf{Der}$$

$$\frac{, [\dagger A \rhd \dagger A] \vdash C}{, [\dagger A] \vdash C}\ \dagger\mathsf{Thread}$$

$$\frac{, \vdash A}{, \vdash \dagger A}\ \dagger\mathcal{R}$$ if , is an independent context with only ! or † formulæ

$$\frac{, [\dagger A] \vdash C}{, [!A] \vdash C}\ \mathsf{!Ser} \qquad \frac{, [!A \otimes !A] \vdash C}{, [!A] \vdash C}\ \mathsf{!Contr}$$

$$\frac{, \vdash A}{, \vdash !A}\ !\mathcal{R}$$ if , is an independent context with only ! formulæ
